

Compactness and an approximation property related to an operator ideal

Anil Kumar Karn[†] and Deba Prasad Sinha[‡]

Abstract

For an operator ideal \mathcal{A} , we study the composition operator ideals $\mathcal{A} \circ \mathcal{K}$, $\mathcal{K} \circ \mathcal{A}$ and $\mathcal{K} \circ \mathcal{A} \circ \mathcal{K}$, where \mathcal{K} is the ideal of compact operators. We introduce a notion of an \mathcal{A} -approximation property on a Banach space and characterise it in terms of the density of finite rank operators in $\mathcal{A} \circ \mathcal{K}$ and $\mathcal{K} \circ \mathcal{A}$.

We propose the notions of ℓ_∞ -extension and ℓ_1 -lifting properties for an operator ideal \mathcal{A} and study $\mathcal{A} \circ \mathcal{K}$, $\mathcal{K} \circ \mathcal{A}$ and the \mathcal{A} -approximation property where \mathcal{A} is injective or surjective and/or with the ℓ_∞ -extension or ℓ_1 -lifting property. In particular, we show that if \mathcal{A} is an injective operator ideal with the ℓ_∞ -extension property, then we have:

(a) X has the \mathcal{A} -approximation property if and only if $(\mathcal{A}^{min})^{inj}(Y, X) = \mathcal{A}^{min}(Y, X)$, for all Banach spaces Y .

(b) The dual space X^* has the \mathcal{A} -approximation property if and only if $((\mathcal{A}^{dual})^{min})^{sur}(X, Y) = (\mathcal{A}^{dual})^{min}(X, Y)$, for all Banach spaces Y . For an operator ideal \mathcal{A} , we study the composition operator ideals $\mathcal{A} \circ \mathcal{K}$,

[†] School of Mathematical Sciences, National Institute of Science Education and Research, Institute of Physics Campus, P.O. Sainik School, Bhubaneswar-751005, India; e-mail: anilkarn@niser.ac.in

[‡] (Lately at) Department of Mathematics, Dyal Singh College (University of Delhi), Lodi Road, New Delhi 110003, India; e-mail: sinha.deba@gmail.com

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1 Introduction

It is well known that a Banach space Y has the approximation property if and only if, $\overline{\mathcal{F}}(X, Y) = \mathcal{K}(X, Y)$ for all Banach spaces X . Similarly, the dual X^* of a Banach space X has the approximation property if and only if $\overline{\mathcal{F}}(X, Y) = \mathcal{K}(X, Y)$ for all Banach spaces Y . However, in general for a pair of Banach spaces X and Y , $\overline{\mathcal{F}}(X, Y) \neq \mathcal{K}(X, Y)$, whereas $\overline{\mathcal{F}}(X, Y)^{inj} = \mathcal{K}(X, Y) = \overline{\mathcal{F}}(X, Y)^{sur}$. In the language of operator ideals $\overline{\mathcal{F}}^{inj} = \mathcal{K} = \overline{\mathcal{F}}^{sur}$. In this language it may be stated that a Banach space X has the approximation property if and only if

$$\overline{\mathcal{F}}^{inj}(Y, X) = \overline{\mathcal{F}}(Y, X)$$

for all Banach spaces Y and the dual space X^* has the approximation property if and only if

$$\overline{\mathcal{F}}^{sur}(X, Y) = \overline{\mathcal{F}}(X, Y)$$

for all Banach spaces Y .

In the papers [7, 8], the authors introduced a class of operator ideals \mathcal{K}_p , ($1 \leq p < \infty$) of compact operators whose adjoints factor through specific subspaces of l_p and showed that $(\Pi_p^{min})^{inj} = \mathcal{K}_p^{dual} = \Pi_p \circ \mathcal{K}$ and $((\Pi_p^{dual})^{min})^{sur} = \mathcal{K}_p^{dual} = \mathcal{K} \circ \Pi_p^{dual}$, where Π_p is the operator ideal of p -summing operators. (In the limiting case $\mathcal{B} \circ \mathcal{K} = \mathcal{K} \circ \mathcal{B} = \mathcal{K}$ holds trivially and also we have $\mathcal{B}^{min} = \overline{\mathcal{F}}$.) Further, in [8] they introduced a notion of the approximation property of *type* p (related to the operator ideal Π_p) and proved that a Banach space X has the approximation property of *type* p if and only if

$$(\Pi_p^{min})^{inj}(Y, X) = \Pi_p^{min}(Y, X)$$

for all Banach spaces Y . Similarly, the dual space X^* has the approximation property of *type* p if and only if

$$((\Pi_p^{dual})^{min})^{sur}(X, Y) = (\Pi_p^{dual})^{min}(X, Y)$$

for all Banach spaces Y .

The above discussions brings to sharp relief the need for the following:

(I) To study the composition operator ideals $\mathcal{A} \circ \mathcal{K}$ and $\mathcal{K} \circ \mathcal{A}$, and also, in general, $\mathcal{K} \circ \mathcal{A} \circ \mathcal{K}$ which we denote by \mathcal{A}^{com} as the compact level objects related to an operator ideal \mathcal{A} , and

(II) To introduce an approximation property related to an operator ideal \mathcal{A} so as to extend the above mentioned characterizations of the approximation property to the operator ideal setting; namely, to study the density of finite rank operators in the relevant compact level of the operator ideal in the corresponding ideal norm.

These are the twin objectives of this paper.

In Section 2, we introduce the kernel procedure $com : \mathcal{A} \rightarrow \mathcal{K} \circ \mathcal{A} \circ \mathcal{K}$ for a general operator ideal \mathcal{A} . We discuss the interplay of this procedure with the standard procedures of operator ideal theory.

The next two sections are mainly preparatory in nature. Section 3 is of independent interest wherein we look closely at the definitions of injective and surjective operator ideals. We show that the composition of two injective (surjective) operator ideals is again injective (respectively, surjective).

In Section 4 we introduce two properties, namely, the l_∞ -extension and the l_1 -lifting properties of operator ideals. These are weaker than the extension and lifting properties that characterise left and right projective operator ideals [1, Ex. 20.6]. We show that many injective and surjective operator ideals possess these properties. We prove that if \mathcal{A}_1 has the l_∞ -extension property, or if \mathcal{A}_2 is injective, then $(\mathcal{A}_1 \circ \mathcal{A}_2)^{inj} = \mathcal{A}_1^{inj} \circ \mathcal{A}_2^{inj}$ and a similar result involving the surjective hull and the l_1 -lifting property is also proved. We show that the composition of two operator ideals both having the l_∞ -extension (l_1 -lifting) property also has the same property. The duality of these two properties for an operator ideal \mathcal{A} and its dual \mathcal{A}^{dual} is also studied.

In Section 5 we investigate the interplay of the kernel procedure ‘ com ’ with the hull procedures ‘ inj ’ and ‘ sur ’. We show that if \mathcal{A} is left accessible, then

$$(\mathcal{A}^{com})^{inj} = (\mathcal{A}^{min})^{inj} = \mathcal{K} \circ \mathcal{A}^{inj}.$$

On the other hand if \mathcal{A} is right accessible and has the l_∞ -extension property, then we have

$$(\mathcal{A}^{com})^{inj} = (\mathcal{A}^{min})^{inj} = \mathcal{A}^{inj} \circ \mathcal{K}.$$

Dual results for surjective operator ideals and for operator ideals with the l_1 -lifting property are also obtained.

In Section 6, we introduce a notion of an approximation property related to an operator ideal \mathcal{A} , namely the \mathcal{A} -approximation property and prove that a Banach space X has the \mathcal{A} -approximation property if and only if

$$\overline{\mathcal{F}(X, Y)}^{\alpha_\kappa} = \mathcal{A} \circ \mathcal{K}(X, Y),$$

where α_κ is the composition ideal norm of $\mathcal{A} \circ \mathcal{K}$. Similarly, the dual X^* has the \mathcal{A} -approximation property if and only if

$$\overline{\mathcal{F}(X, Y)}^{\kappa\alpha^d} = \mathcal{K} \circ \mathcal{A}^{dual}(X, Y),$$

(under certain conditions on \mathcal{A}), where $\kappa\alpha^d$ is the composition ideal norm of $\mathcal{K} \circ \mathcal{A}^{dual}$.

At the end of the section we study the \mathcal{A} -approximation property on a Banach space when \mathcal{A} is injective with the ℓ_∞ -extension property or surjective. In fact, we show that if \mathcal{A} is left accessible injective operator ideal with the ℓ_∞ -extension property, then a Banach space X has the \mathcal{A} -approximation property if and only if

$$(\mathcal{A}^{min})^{inj}(Y, X) = \mathcal{A}^{min}(Y, X),$$

for all Banach spaces Y . The dual space X^* has the \mathcal{A} -approximation property if and only if

$$((\mathcal{A}^{dual})^{min})^{sur}(X, Y) = (\mathcal{A}^{dual})^{min}(X, Y),$$

for all Banach spaces Y .

We also show that if \mathcal{A} is a right accessible, surjective operator ideal, then X has the \mathcal{A} -approximation property if and only if

$$(\mathcal{A}^{min})^{sur}(Y, X) = \mathcal{A}^{min}(Y, X),$$

for all Banach spaces Y . The dual space X^* has the \mathcal{A} -approximation property if and only if

$$((\mathcal{A}^{dual})^{min})^{inj}(X, Y) = (\mathcal{A}^{dual})^{min}(X, Y),$$

for all Banach spaces Y .

Before we close the section we take a quick look at some notations. For a Banach space X , $\kappa_X : X \hookrightarrow X^{**}$ is the natural canonical embedding, and

in case X is complemented in its bidual X^{**} , $P_X : X^{**} \rightarrow X$ shall denote the resulting projection. For the closed unit ball B_X of X , $q_X : l_1(B_X) \rightarrow X$ denotes a usual quotient map and $i_X : X \hookrightarrow l_\infty(B_{X^*})$ is the natural Aloughlu embedding. We denote by \mathcal{B} , \mathcal{K} , $\overline{\mathcal{F}}$ and by Π_p , \mathcal{I}_p and \mathcal{N}_p , for $1 \leq p \leq \infty$, the operator ideals of bounded, compact, approximable, p -summing, p -integral and p -nuclear operators respectively.

We have avoided routine discussions on norms of operator ideals and that of their compositions at several places in the body of the paper.

2 The procedure ‘com’

We begin by formally assigning a symbol to $\mathcal{K} \circ \mathcal{A} \circ \mathcal{K}$ for an operator ideal \mathcal{A} .

Definition 2.1 *Let \mathcal{A} be any quasi normed operator ideal. The composition quasi normed operator ideal $\mathcal{K} \circ \mathcal{A} \circ \mathcal{K}$ shall be denoted by \mathcal{A}^{com} , where \mathcal{K} is the operator ideal of compact operators.*

It is easy to note that the procedure $com : \mathcal{A} \rightarrow \mathcal{A}^{com}$ is monotone. Also as \mathcal{K} is an idempotent ideal, com is an idempotent and a kernel procedure. In this section we discuss some basic facts regarding the procedure com as well as its interplay with other important procedures in the theory of operator ideals .

Let us recall the definition of a minimal kernel \mathcal{A}^{min} of an operator ideal \mathcal{A} as the composition operator ideal $\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}}$, and the corresponding procedure $min : \mathcal{A} \rightarrow \mathcal{A}^{min}$ [6, Section 4.8]. To begin with we consider the interplay of min and com . Since, $\overline{\mathcal{F}} \circ \mathcal{K} = \mathcal{K} \circ \overline{\mathcal{F}} = \overline{\mathcal{F}}$, we have

Proposition 2.2 $(\mathcal{A}^{min})^{com} = (\mathcal{A}^{com})^{min} = \mathcal{A}^{min}$.

Remark 1 Though $\mathcal{A}^{min} \subset \mathcal{A}^{com}$, these are not equal in general. For instance, if \mathcal{B} is the ideal of all bounded linear operators, then $\mathcal{B}^{com} = \mathcal{K}$ and $\mathcal{B}^{min} = \overline{\mathcal{F}}$ and $\mathcal{K} \neq \overline{\mathcal{F}}$. However, in many cases $\mathcal{A}^{min} = \mathcal{A}^{com}$ and we shall come across some such cases in the paper.

Remark 2 We recall the definition of the maximal hull \mathcal{A}^{max} of an operator ideal \mathcal{A} and the corresponding procedure $max : \mathcal{A} \rightarrow \mathcal{A}^{max}$ [6, Section 4.9]. Since $\mathcal{A}^{min} \subset \mathcal{A}^{com} \subset \mathcal{A}$ and since $(\mathcal{A}^{min})^{max} = \mathcal{A}^{max}$, we may conclude that

$(\mathcal{A}^{com})^{max} = \mathcal{A}^{max}$. Also as $\mathcal{A} \subset \mathcal{A}^{max}$, we get $\mathcal{A}^{com} \subset (\mathcal{A}^{max})^{com}$. However, these in general are not equal; for $\mathcal{A} = \overline{\mathcal{F}}$, we get $\overline{\mathcal{F}}^{com} = \overline{\mathcal{F}}$ (for $\overline{\mathcal{F}} = \mathcal{A}^{min}$) and $(\overline{\mathcal{F}}^{max})^{com} = \mathcal{B}^{com} = \mathcal{K}$ so that $\overline{\mathcal{F}}^{com} \neq (\overline{\mathcal{F}}^{max})^{com}$.

Let (\mathcal{A}, α) be a quasi-Banach operator ideal. For Banach spaces X, Y , an operator $T \in \mathcal{B}(X, Y)$ is said to be in $\mathcal{A}^{reg}(X, Y)$, if $\kappa_Y \circ T \in \mathcal{B}(X, Y^{**})$ and $\alpha^{reg}(T) := \alpha(\kappa_Y \circ T)$ [6, Section 4.5]. Now, $(\mathcal{A}^{reg}, \alpha^{reg})$ is a quasi-Banach operator ideal containing \mathcal{A} , the procedure $\mathcal{A} \rightarrow \mathcal{A}^{reg}$ is a hull procedure and \mathcal{A}^{reg} is called the *regular hull* of \mathcal{A} . We now consider the interplay of *reg* and *com*.

Proposition 2.3 $\mathcal{K} \circ (\mathcal{A}^{reg}) = \mathcal{K} \circ \mathcal{A}$. In particular, $(\mathcal{A}^{reg})^{com} = \mathcal{A}^{com}$.

Proof. First note that $\mathcal{A} \subset \mathcal{A}^{reg}$, so that $\mathcal{K} \circ \mathcal{A} \subset \mathcal{K} \circ \mathcal{A}^{reg}$. Next, let $T \in \mathcal{K} \circ \mathcal{A}^{reg}(X, Y)$ for some Banach spaces X and Y . Then there is a Banach space Z and operators $U \in \mathcal{K}(Z, Y)$ and $S \in \mathcal{A}^{reg}(X, Z)$ such that $T = U \circ S$. Thus $\kappa_Z S \in \mathcal{A}(X, Z^{**})$. As U is compact we get that U^{**} is compact with $U^{**}(Z^{**}) \subset \kappa_Y(Y)$. Thus we can find $V \in \mathcal{K}(Z^{**}, Y)$ such that $\kappa_Y \circ V = U^{**}$. Since κ_Y is an isometry and $\kappa_Y \circ U = U^{**} \circ \kappa_Z = \kappa_Y \circ V \circ \kappa_Z$, we may conclude that $U = V \circ \kappa_Z$. Now it follows that $T = U \circ S = V \circ \kappa_Z \circ S \in \mathcal{K} \circ \mathcal{A}(X, Y)$, so that $\mathcal{K} \circ (\mathcal{A}^{reg}) = \mathcal{K} \circ \mathcal{A}$. Consequently, $(\mathcal{A}^{reg})^{com} = \mathcal{A}^{com}$. \triangle

Remark 1 Trivially, $\mathcal{A}^{com} \subset (\mathcal{A}^{com})^{reg}$. However, $\mathcal{A}^{com} \neq (\mathcal{A}^{com})^{reg}$. In fact, for $\mathcal{A} = \mathcal{I}$, the ideal of all integral operators, we have $\mathcal{I}^{com} = \mathcal{N}$, the ideal of all nuclear operators; and $(\mathcal{I}^{com})^{reg} = \mathcal{N}^{reg}$, but $\mathcal{I}^{com} = \mathcal{N}$ is not regular.

Remark 2 A striking difference in the behaviour of $\overline{\mathcal{F}}$ and \mathcal{K} in composition is now evident. Indeed, $\overline{\mathcal{F}} \circ \mathcal{I} = \overline{\mathcal{F}} \circ \mathcal{I} \circ \overline{\mathcal{F}} = \mathcal{N} = \mathcal{I} \circ \overline{\mathcal{F}}$, but as \mathcal{N} is not regular, $\mathcal{K} \circ \mathcal{I} = \mathcal{K} \circ \mathcal{I} \circ \mathcal{K} = \mathcal{N} \neq \mathcal{N}^{reg} = \mathcal{I} \circ \mathcal{K}$ [5, Theorem 2.1].

Let (\mathcal{A}, α) be a quasinormed operator ideal. For Banach spaces X, Y and operator $T \in \mathcal{B}(X, Y)$ is said to be in $\mathcal{A}^{dual}(X, Y)$ if $T^* \in \mathcal{A}(Y^*, X^*)$ and $\alpha^{dual}(T) = \alpha(T^*)$ [6, Section 4.4]. Now, $(\mathcal{A}^{dual}, \alpha^{dual})$ is a quasinormed operator ideal and is called the *dual* of \mathcal{A} . The following observation, which are routine in nature, shall be frequently used in the paper: For an operator ideal \mathcal{A} we have

1. $\mathcal{A} \cup \mathcal{A}^{dualdual} \subset \mathcal{A}^{reg}$; and
2. $\mathcal{A} \subset \mathcal{A}^{dualdual}$ if and only if $\mathcal{A}^{dualdual} = \mathcal{A}^{reg}$.

Next, we study the interplay of the procedures *com* and *dual*. We note that for any operator ideal \mathcal{A} , we have

$$(\mathcal{A}^{dual})^{com} \subset (\mathcal{A}^{com})^{dual}.$$

But in general, these are not equal; for instance, the ideal of \mathcal{I} of integral operators, $\mathcal{I} = \mathcal{I}^{dual}$ and $(\mathcal{I}^{dual})^{com} = \mathcal{I}^{com} = \mathcal{N} \neq \mathcal{N}^{dual} = (\mathcal{I}^{com})^{dual}$. However,

Proposition 2.4 *For an operator ideal \mathcal{A} , with $\mathcal{A} \subset \mathcal{A}^{dualdual}$, we have*

$$(\mathcal{A}^{com})^{dual} = ((\mathcal{A}^{dual})^{com})^{reg}.$$

Proof. Let $T \in (\mathcal{A}^{com})^{dual}(X, Y)$, for Banach spaces X and Y . Then $T^* \in \mathcal{A}^{com}(Y^*, X^*)$. There exists Banach spaces Z_1, Z_2 and operators $U \in \mathcal{K}(Z_2, X^*)$, $V \in \mathcal{K}(Y^*, Z_1)$, $S \in \mathcal{A}(Z_1, Z_2)$ such that $T^* = U \circ S \circ V$. Since $\mathcal{A} \subset \mathcal{A}^{dualdual}$, we get $S^* \in \mathcal{A}^{dual}(Z_2^*, Z_1^*)$ so that $T^{**} = V^* \circ S^* \circ U^* \in (\mathcal{A}^{dual})^{com}(X^{**}, Y^{**})$. It follows that $\kappa_Y \circ T = T^{**} \circ \kappa_X = V^* \circ S^* \circ U^* \circ \kappa_X \in (\mathcal{A}^{dual})^{com}(X, Y^*)$, whence $T \in ((\mathcal{A}^{dual})^{com})^{reg}$. Now $(\mathcal{A}^{com})^{dual}$ being regular, the result follows. \triangle

3 Injective and surjective operator ideals - A revisit

For every quasinormed operator ideal (\mathcal{A}, α) , there is a smallest injective operator ideal \mathcal{A}^{inj} containing \mathcal{A} . For Banach spaces X, Y and the natural Alouglu embedding $i_Y : Y \hookrightarrow l_\infty(B_{Y^*})$, an operator $T \in \mathcal{B}(X, Y)$ is in \mathcal{A}^{inj} if and only if $i_Y \circ T \in \mathcal{A}$ with $\alpha^{inj}(T) := \alpha(i_Y \circ T)$. The ideal $(\mathcal{A}^{inj}, \alpha^{inj})$ is a quasinormed operator ideal [6, Section 4.6]. The procedure $\mathcal{A} \rightarrow \mathcal{A}^{inj}$ is a hull procedure and \mathcal{A}^{inj} is called the injective hull of \mathcal{A} . The operator ideal \mathcal{A} is said to be injective if $\mathcal{A}^{inj} = \mathcal{A}$.

First of all we give an interesting and useful characterization of injective operator ideals.

Definition 3.1 *An operator ideal \mathcal{A} is said to have the restricted range property (RRP, for short), if for arbitrary Banach spaces X, Y and $T \in \mathcal{A}(X, Y)$, we have $\hat{T} \in \mathcal{A}(X, \overline{R(X)})$. Here $R(X)$ is the range of T in Y and $\hat{T}(x) = T(x)$ for all $x \in X$.*

Lemma 3.2 *An operator ideal \mathcal{A} is injective if and only if it has the restricted range property.*

Proof. It follows from [6, Proposition 8.5.4] that an injective operator ideal has the *RRP*.

Conversely, let \mathcal{A} have the *RRP*. Let $T \in \mathcal{A}^{inj}(X, Y)$ for some Banach spaces. Then $i_Y \circ T \in \mathcal{A}(X, l_\infty(B_{Y^*}))$. Since \mathcal{A} has the *RRP*, we have $T_1 \in \mathcal{A}(X, \overline{i_Y \circ T(X)})$, where $T_1(x) = i_Y \circ T(x)$, $x \in X$. As i_Y is an isometry and as $\overline{T(X)} \subset Y$ we may obtain an isometry $I_0 : \mathcal{B}(\overline{i_Y \circ T(X)}, Y)$ given by $I_0(i_Y \circ T(x)) = Tx$, $x \in X$. Now $I_0 \circ T_1(x) = I_0(i_Y \circ T(x)) = T(x)$ for all $x \in X$. In other words, $T = I_0 \circ T_1 \in \mathcal{A}(X, Y)$ so that \mathcal{A} is injective. \triangle

Theorem 3.3 *If \mathcal{A}_1 and \mathcal{A}_2 are injective operator ideals, then so is $\mathcal{A}_1 \circ \mathcal{A}_2$.*

Proof. Let $T \in (\mathcal{A}_1 \circ \mathcal{A}_2)^{inj}(X, Y)$ and let $I : Y \rightarrow Z$ be an isometry for some Banach spaces X, Y and Z . Then $I \circ T \in (\mathcal{A}_1 \circ \mathcal{A}_2)(X, Z)$. Thus there exists a Banach space W and operators $U \in \mathcal{A}_1(W, Z)$ and $V \in \mathcal{A}_2(X, W)$ such that $I \circ T = U \circ V$. Since \mathcal{A}_2 being injective has the *RRP*, by using the ideal property of \mathcal{A}_1 we may assume that $W = \overline{V(X)}$. Next, we set $U_0 : V(X) \rightarrow Y$ by $U_0(Vx) = Tx$, for all $X \in X$. If $Vx_0 = 0$ for some $x_0 \in X$, then $I \circ T(x_0) = U \circ V(x_0) = 0$. Since I is an isometry we conclude that $Tx = 0$. Thus U_0 is a well defined linear map. Also

$$\|U_0(Vx)\| = \|Tx\| = \|I \circ (x)\| = \|U \circ V(x)\| \leq \|U\| \|Vx\|,$$

for all $x \in X$, so that $U_0 \in \mathcal{B}(V(X), Y)$. Thus we can extend U_0 to $U_1 \in \mathcal{B}(W, Y)$. Now

$$I \circ U_1(Vx) = I \circ T(x) = U(Vx), \text{ for all } x \in X$$

so that $U = I \circ U_1 \in \mathcal{A}_1(W, Z)$. As \mathcal{A}_1 is injective, we conclude that $U_1 \in \mathcal{A}_1(W, Z)$. It follows that $T = U_1 \circ V \in (\mathcal{A}_1 \circ \mathcal{A}_2)(X, Y)$. In other words $\mathcal{A}_1 \circ \mathcal{A}_2$ is injective. \triangle

Remark If \mathcal{A} is an injective operator ideal then so are $\mathcal{K} \circ \mathcal{A}$ and $\mathcal{A} \circ \mathcal{K}$. In particular, \mathcal{A}^{com} is injective whenever \mathcal{A} is so.

For every quasinormed operator ideal (\mathcal{A}, α) , there is a smallest operator ideal \mathcal{A}^{sur} containing \mathcal{A} . For Banach spaces X and Y , an operator $T \in \mathcal{B}(X, Y)$ is in \mathcal{A}^{sur} if and only if $T \circ q_X \in \mathcal{A}$ with $\alpha^{sur}(T) := \alpha(T \circ q_X)$.

The ideal $(\mathcal{A}^{sur}, \alpha^{sur})$ is a quasinormed operator ideal [6, Section 4.7]. The procedure $\mathcal{A} \rightarrow \mathcal{A}^{sur}$ is a hull procedure and \mathcal{A}^{sur} is called the surjective hull of \mathcal{A} . The operator ideal \mathcal{A} is called surjective if $\mathcal{A}^{sur} = \mathcal{A}$. We can dualise the above ideas for surjective operator ideals as follows.

To prove this theorem we shall use the following characterization of surjective operator ideals.

Definition 3.4 *An operator ideal \mathcal{A} is said to have the quotiented domain property (QDP, for short) if for arbitrary Banach spaces X, Y and operator $T \in \mathcal{A}(X, Y)$, we have $T_0 \in \mathcal{A}(X/Ker\ T, Y)$ is the kernel of T and $T_0(X + Ker\ T) = Tx$, for all $x \in X$.*

Lemma 3.5 *An operator ideal \mathcal{A} is surjective if and only if it has quotiented domain property.*

Proof. It follows from [6, proposition 8.5.4] that a surjective operator ideal has the QDP.

Conversely, assume that \mathcal{A} has the QDP and for a pair of Banach spaces X and Y , let $T \in \mathcal{A}^{sur}(X, Y)$. Then $T \circ q_X \in \mathcal{A}(l_1(B_X), Y)$. Let $Z = l_1(B_X)/Ker(T \circ q_X)$. Let $T_0 : Z \rightarrow Y$ be the map corresponding to $T \circ q_X$ and let $q : l_1(B_X) \rightarrow Z$ be the natural quotient map. Then $T \circ q_X = T_0 \circ q$, and we have $T_0 \in \mathcal{A}(Z, Y)$ for \mathcal{A} has the QDP. For each $x \in X$ we set $V_0(x) = q(\alpha)$, where $x = q_X(\alpha)$. Now if $q_X(\alpha) = 0$, then $Tq_X(\alpha) = 0$ so that $\alpha \in Ker(T \circ q_X)$. Thus $V_0(0) = q(0) = 0$. Therefore, $V_0 : X \rightarrow Z$ is a well defined linear operator. Also, for any $x \in X$ we have

$$\|V_0(x)\| = \inf\{\|q_X(\alpha)\| : x = q_X(\alpha)\} \leq \inf\{\|\alpha\| : x = q_X(\alpha)\} = \|x\|.$$

Thus $V_0 \in \mathcal{B}(X, Z)$. Now $T_0 \circ V_0(x) = T_0 \circ q(\alpha) = T \circ q_X(\alpha) = T(x)$ for all $x \in X$, so that $T = T_0 \circ V_0 \in \mathcal{A}(X, Y)$. Hence \mathcal{A} is surjective. \triangle

Theorem 3.6 *If \mathcal{A}_1 and \mathcal{A}_2 are surjective operator ideals, then so is $\mathcal{A}_1 \circ \mathcal{A}_2$.*

Proof. Let X, Y be Banach spaces and $T \in (\mathcal{A}_1 \circ \mathcal{A}_2)^{sur}(X, Y)$. Then $T \circ q_X \in (\mathcal{A}_1 \circ \mathcal{A}_2)(l_1(B_X), Y)$. Thus there is a Banach space Z and operators $U \in \mathcal{A}_1(Z, Y)$ and $V \in \mathcal{A}_2(l_1(B_X), Z)$ such that $T \circ q_X = U \circ V$. Let $q_U : Z \rightarrow Z/Ker\ U$ be the natural quotient map and $U_0 : Z/Ker\ U \rightarrow Y$ be the map corresponding to U . Then, $U = U_0 \circ q_U$. Since \mathcal{A} has the QDP, we have $U_0 \in \mathcal{A}(Z/Ker\ U, Y)$. For $x \in X$ we set $V_0(x) = q_U V(\alpha)$, where $x = q_X(\alpha)$.

Then, proceeding as in Lemma 3.5, we may obtain $V_0 \in \mathcal{B}(X, Z/\text{Ker}U)$ and that $V_0 \circ q_X = q_U \circ V \in \mathcal{A}_2(l_1(B_X), Z/\text{Ker}U)$. Since \mathcal{A}_2 is surjective, we conclude further that $V_0 \in \mathcal{A}_2(X, Z/\text{Ker}U)$. Now for all $x \in X$

$$U_0 \circ V_0(x) = U_0(q_U \circ V(\alpha)) = U \circ V(\alpha) = T \circ q_X(\alpha) = T(x),$$

so that $T = U_0 \circ V_0 \in \mathcal{A}_1 \circ \mathcal{A}_2(X, Y)$. Hence $\mathcal{A}_1 \circ \mathcal{A}_2$ is surjective. \triangle

Remark Let \mathcal{A} be a surjective operator ideal, then so are $\mathcal{K} \circ \mathcal{A}$ and $\mathcal{A} \circ \mathcal{K}$. In particular, \mathcal{A}^{com} is surjective if \mathcal{A} is so.

4 Extension and lifting properties for operator ideals

Let \mathcal{A}_1 and \mathcal{A}_2 be two operator ideals. Then by Theorem 3.3, $\mathcal{A}_1^{inj} \circ \mathcal{A}_2^{inj}$ is an injective operator ideal containing $(\mathcal{A}_1 \circ \mathcal{A}_2)^{inj}$. Also it follows from the definition of injectivity, that $\mathcal{A}_1^{inj} \circ \mathcal{A}_2 \subset (\mathcal{A}_1 \circ \mathcal{A}_2)^{inj}$. Thus

$$\mathcal{A}_1^{inj} \circ \mathcal{A}_2 \subset (\mathcal{A}_1 \circ \mathcal{A}_2)^{inj} \subset \mathcal{A}_1^{inj} \circ \mathcal{A}_2^{inj}.$$

In particular, when \mathcal{A}_2 is injective, we have

$$(\mathcal{A}_1 \circ \mathcal{A}_2)^{inj} = \mathcal{A}_1^{inj} \circ \mathcal{A}_2.$$

Next, we propose a condition on \mathcal{A}_1 for which $(\mathcal{A}_1 \circ \mathcal{A}_2)^{inj} = \mathcal{A}_1^{inj} \circ \mathcal{A}_2^{inj}$.

Definition 4.1 *An operator ideal (\mathcal{A}, α) is said to have the l_∞ -extension property if for a Banach spaces X , a set Γ and an operator $T \in \mathcal{A}(X, l_\infty(\Gamma))$, there is an operator $\tilde{T} \in \mathcal{A}(l_\infty(B_{X^*}), l_\infty(\Gamma))$ such that $T = \tilde{T} \circ i_X$, with $\alpha(\tilde{T}) = \alpha(T)$, where $i_X : X \hookrightarrow l_\infty(B_{X^*})$ is the Alouglu embedding.*

Since $l_\infty(\Gamma)$ spaces are injective, the ideals \mathcal{B} and \mathcal{K} of all bounded and all compact operators respectively, have the l_∞ -extension property. Since $l_\infty(\Gamma)$ has the approximation property, $\overline{\mathcal{F}}(X, l_\infty(\Gamma)) = \mathcal{K}(X, l_\infty(\Gamma))$ for all Banach spaces X . It follows that $\overline{\mathcal{F}}$ has the l_∞ -extension property. The ideal \mathcal{I}_p of p -integral operators for $1 \leq p \leq \infty$ has this property. In fact, these operator ideals enjoy a much stronger extension property [2, Proposition 6.12]. As for the operator ideals Π_p , $\Pi_p(X, Y)$ coincides with $\mathcal{I}_p(X, Y)$ if Y is an l_∞ -space. It follows that Π_p also enjoys the l_∞ -extension property.

Theorem 4.2 *Let \mathcal{A}_1 and \mathcal{A}_2 be two operator ideals. If \mathcal{A}_1 has the l_∞ -extension property, or if \mathcal{A}_2 is injective, then*

$$(\mathcal{A}_1 \circ \mathcal{A}_2)^{inj} = \mathcal{A}_1^{inj} \circ \mathcal{A}_2^{inj}.$$

Proof. Let X and Y be Banach spaces and let $T \in \mathcal{A}_1^{inj} \circ \mathcal{A}_2^{inj}(X, Y)$. Then there is a Banach space Z and operators $U \in \mathcal{A}_1^{inj}(Z, Y)$ and $V \in \mathcal{A}_2^{inj}(X, Z)$ such that $T = U \circ V$. If \mathcal{A}_1 has the l_∞ -extension property, then there is an $\tilde{U} \in \mathcal{A}_1(l_\infty(B_{Z^*}), l_\infty(B_{Y^*}))$ such that $i_Y \circ U = \tilde{U} \circ i_Z$. Thus we have

$$i_Y \circ T = i_Y \circ U \circ V = \tilde{U} \circ i_Z \circ V = \tilde{U} \circ \tilde{V},$$

where $\tilde{V} = i_Z \circ V \in \mathcal{A}_2(X, l_\infty(B_{Z^*}))$. It follows that $T \in \mathcal{A}_1^{inj} \circ \mathcal{A}_2^{inj}(X, Y)$. The remaining part of the proof follows from the above discussion. \triangle

Proposition 4.3 *If \mathcal{A}_1 and \mathcal{A}_2 are two operators ideals both having the l_∞ -extension property, then so has $\mathcal{A}_1 \circ \mathcal{A}_2$.*

Proof. Let X be a Banach space, Γ a set and consider the operator $T \in \mathcal{A}_1 \circ \mathcal{A}_2(X, l_\infty(\Gamma))$. Then there is a Banach space Y and operators $U \in \mathcal{A}_1(Y, l_\infty(\Gamma))$ and $V \in \mathcal{A}_2(X, Y)$ such that $T = U \circ V$. Since both \mathcal{A}_1 and \mathcal{A}_2 have the l_∞ -extension property, U has an extension $\tilde{U} \in \mathcal{A}_1(l_\infty(B_{Y^*}), l_\infty(\Gamma))$ and $i_Y \circ V$ has an extension $\tilde{V} \in \mathcal{A}_2(l_\infty(B_{X^*}), l_\infty(B_{Y^*}))$. Thus, $\tilde{U} \circ \tilde{V} \circ i_X = \tilde{U} \circ i_Y \circ V = U \circ V$. Hence, $\tilde{U} \circ \tilde{V} \in \mathcal{A}_1 \circ \mathcal{A}_2$ extends T . \triangle

Let \mathcal{A}_1 and \mathcal{A}_2 be two operator ideals. It follows by Theorem 3.6 that

$$\mathcal{A}_1 \circ \mathcal{A}_2^{sur} \subset (\mathcal{A}_1 \circ \mathcal{A}_2)^{sur} \subset \mathcal{A}_1^{sur} \circ \mathcal{A}_2^{sur}.$$

In particular, if \mathcal{A}_1 is surjective then $(\mathcal{A}_1 \circ \mathcal{A}_2)^{sur} = \mathcal{A}_1 \circ \mathcal{A}_2^{sur}$. We propose, once more, a condition on \mathcal{A}_2 such that $(\mathcal{A}_1 \circ \mathcal{A}_2)^{sur} = \mathcal{A}_1^{sur} \circ \mathcal{A}_2^{sur}$.

Definition 4.4 *An operator ideal (\mathcal{A}, α) is said to have the l_1 -lifting property if for any set Γ , a Banach space Y and an operator $T \in \mathcal{A}(l_1(\Gamma), Y)$, there is an $\hat{T} \in (l_1(\Gamma), l_1(B_Y))$ such that $q_Y \circ \hat{T} = \kappa_Y \circ T$, with $\alpha(\hat{T}) = \alpha(T)$, where $\kappa_Y : Y \hookrightarrow Y^{**}$ is the canonical embedding and $q_Y : l_1(B_Y) \rightarrow Y$ is the natural quotient map.*

The lifting property of $l_1(\Gamma)$ ensures that both the operator ideals \mathcal{B} and \mathcal{K} have the l_1 -lifting property. Since the $l_1^*(\Gamma)$ has the approximation property, $\overline{\mathcal{F}}(l_1(\Gamma), Y) = \mathcal{K}(l_1(\Gamma), Y)$ for all Banach spaces Y , it follows that $\overline{\mathcal{F}}$ has the l_1 -lifting property. Using the l_∞ -extension property of Π_p it is easy to verify that Π_p^{dual} also has the l_1 -lifting property. Dualising the proof of Theorem 4.2, we can prove the following.

Theorem 4.5 *Let \mathcal{A}_1 and \mathcal{A}_2 be two operator ideals. If \mathcal{A}_1 is surjective, or if \mathcal{A}_2 has the l_1 -lifting property, then*

$$(\mathcal{A}_1 \circ \mathcal{A}_2)^{sur} = \mathcal{A}_1^{sur} \circ \mathcal{A}_2^{sur}.$$

Proposition 4.6 *If \mathcal{A}_1 and \mathcal{A}_2 are two operators ideals both having the l_1 -lifting property, then so has $\mathcal{A}_1 \circ \mathcal{A}_2$.*

Proof. Let Y be a Banach space, Γ a set and $T \in \mathcal{A}_1 \circ \mathcal{A}_2(l_1(\Gamma), Y)$. Then, there is a Banach space Z and operators $U \in \mathcal{A}_1(X, Y)$ and $V \in \mathcal{A}_2(l_1(\Gamma), Z)$ such that $T = U \circ V$. Since, \mathcal{A}_2 has the l_1 -lifting property, there is an operator $\hat{V} \in \mathcal{A}_2(l_1(\Gamma), l_1(B_Z))$ such that $V = q_Z \circ \hat{V}$. Next, $U \circ q_Z \in \mathcal{A}_1(l_1(B_Z), Y)$ and \mathcal{A}_1 has the l_1 -lifting property. Thus there exists an operator $\hat{U} \in \mathcal{A}_1(l_1(B_Z), l_1(B_Y))$ such that $U \circ q_Z = q_Y \circ \hat{U}$. Thus

$$T = U \circ V = U \circ q_Z \circ \hat{V} = q_Y \circ \hat{U} \circ \hat{V}.$$

Now, $\hat{U} \circ \hat{V} \in \mathcal{A}_1 \circ \mathcal{A}_2(l_1(\Gamma), l_1(B_Y))$, so that we may conclude that $\mathcal{A}_1 \circ \mathcal{A}_2$ has the l_1 -lifting property. \triangle

Now we investigate the duality relationship between the l_∞ -extension and the l_1 -lifting properties. To begin with we prove two lemmas.

Lemma 4.7 *Let X be a Banach space, Γ a set and $T \in \mathcal{A}(X, l_\infty(\Gamma))$. Then the following are equivalent:*

- (a) *For any Banach space Y , any isometry $I : X \rightarrow Y$ and $\epsilon > 0$, there is a $\tilde{T} \in \mathcal{A}(Y, l_\infty(\Gamma))$ with $\alpha(\tilde{T}) \leq (1 + \epsilon)\alpha(T)$ such that $T = \tilde{T} \circ I$.*
- (b) *For some set Ω , and isometry $i_0 : X \rightarrow l_\infty(\Omega)$ and $\epsilon > 0$, there is a $T_0 \in \mathcal{A}(l_\infty(\Omega), l_\infty(\Gamma))$ with $\alpha(\tilde{T}) \leq (1 + \epsilon)\alpha(T)$ such that $T = T_0 \circ i_0$.*

Proof. It suffices to show that (b) \Rightarrow (a). Let $I : X \rightarrow Y$ be any isometry. Consider the canonical embedding $i_X : X \hookrightarrow l_\infty(B_{X^*})$. Since $l_\infty(B_{X^*})$ has the extension property, there is a bounded operator $\tilde{i}_X : Y \rightarrow l_\infty(B_{X^*})$ with $\|\tilde{i}_X\| \leq (1 + \epsilon/4)$ such that $i_X = \tilde{i}_X \circ I$. Now by (b), T has an extension $T_0 : l_\infty(B_{X^*}) \rightarrow l_\infty(\Gamma)$ with $\alpha(T_0) < (1 + \epsilon/4)\alpha(T)$ such that $T = T_0 \circ i_X = \tilde{T} \circ I$ where $\tilde{T} = T_0 \circ \tilde{i}_X \in \mathcal{A}(Y, l_\infty(\Gamma))$. \triangle

Lemma 4.8 *Let Y be a Banach space, Γ a set and $T \in \mathcal{A}(l_1(\Gamma), Y)$. Then the following are equivalent:*

- (a) For any Banach space X , metric quotient $Q : X \rightarrow Y$ and $\epsilon > 0$, there is a $\hat{T} \in \mathcal{A}((l_1(\Gamma), Y))$ with $\alpha(\hat{T}) \leq (1 + \epsilon)\alpha(T)$ such that $T = Q \circ \hat{T}$
- (b) For some set Ω , a metric quotient $q_0 : l_1(\Omega) \rightarrow Y$ and $\epsilon > 0$, there is a $T_0 \in \mathcal{A}(l_1(\Gamma), l_1(\Omega))$ with $\alpha(T_0) \leq (1 + \epsilon)\alpha(T)$ such that $T = q_0 \circ T_0$.

Proof. It suffices to prove that (b) \Rightarrow (a). Let $Q : X \rightarrow Y$ be any metric quotient. Consider the natural metric quotient $q_X : l_1(B_X) \rightarrow X$. Since $l_1(B_X)$ has the lifting property, there is a bounded operator $\hat{q}_X : l_1(B_X) \rightarrow Y$ with $\|\hat{q}_X\| < 1 + \epsilon/4$ such that $\hat{q}_X = Q \circ q_X$. Now by (b), T has a lifting $T_0 : \mathcal{A}(l_1(\Gamma), l_1(B_X))$ with $\alpha(T_0) < (1 + \epsilon/4)\alpha(T)$ such that $T = \hat{q}_X \circ T_0 = Q \circ \hat{T}$ where $\hat{T} = q_X \circ T_0 \in \mathcal{A}(l_1(\Gamma), X)$. \triangle

Remark Since $\mathcal{A}(X, l_\infty(\Gamma)) = \mathcal{A}^{inj}(X, l_\infty(\Gamma)) = \mathcal{A}^{reg}(X, l_\infty(\Gamma))$, we conclude that \mathcal{A} has the l_∞ -extension property if and only if \mathcal{A}^{inj} has the l_∞ -extension property if and only if \mathcal{A}^{reg} has the l_∞ -extension property. Dually, since $\mathcal{A}(l_1(\Gamma), Y) = \mathcal{A}^{sur}(l_1(\Gamma), Y)$, we may further conclude that \mathcal{A} has the l_1 -lifting property if and only if so does \mathcal{A}^{sur} . However, we are not aware of the dual situation for the case of the regular hull.

Theorem 4.9 *If \mathcal{A} has the l_∞ -extension property, then \mathcal{A}^{dual} has the l_1 -lifting property. If $\mathcal{A} \subset \mathcal{A}^{dual \ dual}$ (or equivalently, $\mathcal{A}^{dual \ dual} = \mathcal{A}^{reg}$), then the converse also holds*

Proof. Let Y be a Banach space, Γ a set and $T \in \mathcal{A}^{dual}(l_1(\Gamma), Y)$. Then $T^* \in \mathcal{A}(Y^*, l_\infty(\Gamma))$. Since \mathcal{A} has the l_∞ -extension property, there is an $S_0 \in \mathcal{A}(l_\infty(B_{Y^*}), l_\infty(\Gamma))$ such that $T^* = S_0 \circ q_Y^*$. Consider the adjoint $S_0^* : l_1(\Gamma)^{**} \rightarrow l_1(B_{Y^*})^*$ of S_0 , and put $\hat{T} = P_{l_1(B_X)} \circ S_0^* \circ \kappa_{l_1(\Gamma)}$. Then $\hat{T}^* = S_0$, so that $\hat{T} \in \mathcal{A}^{dual}(l_1(\Gamma), l_1(B_{Y^*}))$, and we have $T = q_Y \circ \hat{T}$. Hence \mathcal{A}^{dual} has the l_1 -lifting property.

Conversely, let $\mathcal{A} \subset \mathcal{A}^{dual \ dual}$ and let \mathcal{A}^{dual} have the l_1 -lifting property. For a Banach space X and a set Γ , let $T \in \mathcal{A}(X, l_\infty(\Gamma)) \subset \mathcal{A}^{dual}(X, l_\infty(\Gamma))$. Then $T^* \in \mathcal{A}^{dual}(l_1(\Gamma)^{**}, X^*)$, so that $T^* \circ \kappa_{l_1(\Gamma)} \in \mathcal{A}^{dual}(l_1(\Gamma), X^*)$. Since \mathcal{A}^{dual} has the l_1 -lifting property, there is a $S_0 \in \mathcal{A}^{dual}(l_1(\Gamma), l_1(B_{X^*}))$ such that $T^* \circ \kappa_{l_1(\Gamma)} = q_X \circ S_0$. Thus

$$T = \kappa_{l_1(\Gamma)}^* \circ \kappa_{l_\infty(\Gamma)} \circ T = \kappa_{l_1(\Gamma)}^* \circ T^{**} \circ \kappa_X = S_0^* \circ q_{X^*}^* \circ \kappa_X = S_0^* \circ i_X.$$

Thus, $S_0^* \in \mathcal{A}(l_\infty(B_{X^*}), l_\infty(\Gamma))$. Hence \mathcal{A} has the l_∞ -extension property. \triangle

Theorem 4.10 *Let $\mathcal{A} \subset \mathcal{A}^{dual\ dual}$ (or equivalently, $\mathcal{A}^{dual\ dual} = \mathcal{A}^{reg}$). Then, \mathcal{A} has the l_1 -lifting property if and only if \mathcal{A}^{dual} has the l_∞ -extension property.*

Proof. First assume that \mathcal{A} has the l_1 -lifting property. For a Banach space X and a set Γ , let $T \in \mathcal{A}^{dual}(X, l_\infty(\Gamma))$. Then $T^* \in \mathcal{A}(l_1(\Gamma)^{**}, X^*)$ so that $T^* \circ \kappa_{l_1(\Gamma)} \in \mathcal{A}(l_1(\Gamma), X^*)$. Since \mathcal{A} has the l_1 -lifting property, there is an $S_0 \in \mathcal{A}(l_1(\Gamma), l_1(B_{X^*}))$ such that $T^* \circ \kappa_{l_1(\Gamma)} = q_{X^*} \circ S_0$. Then

$$T = \kappa_{l_1(\Gamma)}^* \circ \kappa_{l_\infty(\Gamma)} \circ T = \kappa_{l_1(\Gamma)}^* \circ T^{**} \circ \kappa_X = S_0^* \circ q_{X^*}^* \circ \kappa_X = S_0^* \circ i_X,$$

for $\kappa_{l_1(\Gamma)}^* = P_{l_\infty(\Gamma)}$, $P_{l_\infty(\Gamma)} \circ \kappa_{l_\infty(\Gamma)} = I_{l_\infty(\Gamma)}$ and $q_{X^*}^* \circ \kappa_X = i_X$. Put $S_0^* = \hat{T}$. Since $S_0 \in \mathcal{A}(l_1(\Gamma), l_\infty(B_{X^*})) \subset \mathcal{A}^{dual\ dual}(l_1(\Gamma), l_1(B_{X^*}))$, we get $\hat{T} \in \mathcal{A}^{dual}(l_\infty(B_{X^*}), l_\infty(\Gamma))$. Thus \mathcal{A}^{dual} has the l_∞ -extension property.

Conversely, assume that \mathcal{A}^{dual} has the l_∞ -extension property. Then by Theorem 4.9, $\mathcal{A}^{dual\ dual}$ has the l_1 -lifting property. Let $T \in \mathcal{A}(l_1(\Gamma), l_1(B_Y)) \subset \mathcal{A}^{dual\ dual}(l_1(\Gamma), l_1(B_Y))$, such that $T = q_Y \circ \hat{T}$. Now $\hat{T}^{**} \in \mathcal{A}(l_1(\Gamma)^{**}, l_1(B_Y)^{**})$ so that $\hat{T} = P_{l_1(B_X)} \circ \kappa_{l_1(B_X)} \circ \hat{T} = P_{l_1(B_X)} \circ \hat{T}^{**} \circ \kappa_{l_1(\Gamma)} \in \mathcal{A}(l_1(\Gamma), l_1(B_Y))$. Therefore, \mathcal{A} has the l_1 -lifting property. \triangle

5 Compact kernels of injective and surjective operator ideals

In this section we record the interplay of the kernel procedure ‘*com*’ with the hull procedures ‘*inj*’ and ‘*sur*’. The last two sections of preparation leads us to several observations.

Proposition 5.1 *Let \mathcal{A} be an operator ideal. The following are in order*

(1) *Since \mathcal{K} is injective, we have*

$$(\mathcal{A} \circ \mathcal{K})^{inj} = \mathcal{A}^{inj} \circ \mathcal{K}.$$

[Theorem 3.3]

(1') *If \mathcal{A} has the l_∞ -extension property, then in view of $\mathcal{K} = \overline{\mathcal{F}}^{inj}$, we have*

$$\mathcal{A}^{inj} \circ \mathcal{K} = (\mathcal{A} \circ \overline{\mathcal{F}})^{inj}.$$

[Theorem 4.2]

(2) Since \mathcal{K} is injective and also has the l_∞ -extension property, we have

$$\mathcal{K} \circ \mathcal{A}^{inj} = (\mathcal{K} \circ \mathcal{A})^{inj}.$$

[Theorem 4.2]

(2') In particular, since $\mathcal{K} = \overline{\mathcal{F}}^{inj}$ and $\overline{\mathcal{F}}$ has the l_∞ -extension property, we have

$$\mathcal{K} \circ \mathcal{A}^{inj} = \overline{\mathcal{F}}^{inj} \circ \mathcal{A}^{inj} = (\overline{\mathcal{F}} \circ \mathcal{A})^{inj}.$$

(3) It follows from 1 and 2 above, that

$$(\mathcal{A}^{com})^{inj} = (\mathcal{A}^{inj})^{com}.$$

(3') If \mathcal{A} has the l_∞ -extension property, then by 1' and 2' above, we have

$$(\mathcal{A}^{inj})^{com} = \mathcal{K} \circ (\mathcal{A} \circ \overline{\mathcal{F}})^{inj} = (\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}})^{inj} = (\mathcal{A}^{min})^{inj}.$$

Remarks 1. Since \mathcal{I} has the l_∞ -extension property for $1 \leq p < \infty$, Proposition 19.2.16 in [6] follows from Proposition 5.1(3') above. Also, note that for the operator ideal \mathcal{K}_p defined in [7], by Proposition 5.1(1) above, we have $\mathcal{K}_p^{dual} = \Pi_p \circ \mathcal{K} = \mathcal{I}_p^{inj} \circ \mathcal{K} = (\mathcal{I}_p \circ \mathcal{K})^{inj} = \mathcal{N}_p^{inj} = \mathcal{QN}_p$, where \mathcal{QN}_p is the operator ideal of p -quasi nuclear operators.

2. Since Π_p is a left accessible injective operator ideal with the l_∞ -extension property Π_p^{dual} is surjective, and by Theorem 4.9 it also has the l_1 -lifting property. Thus the results $(\Pi_p^{min})^{inj} = \mathcal{K}_p^{dual} = \Pi_p \circ \mathcal{K}$ and $((\Pi_p^{dual})^{min})^{sur} = \mathcal{K}_p^{dual\ dual} = \mathcal{K} \circ \Pi_p^{dual}$ for $1 \leq p \leq \infty$, obtained in [8] provide us specific examples of situations as clarified by the above proposition. Also note that in the limiting case $\mathcal{B} \circ \mathcal{K} = \mathcal{K} \circ \mathcal{B} = \mathcal{K}$ holds trivially and also we have $(\mathcal{B}^{min})^{inj} = \overline{\mathcal{F}}^{inj} = \mathcal{K} = (\mathcal{B}^{min})^{sur} = \overline{\mathcal{F}}^{sur}$.

Proposition 5.2 *Let \mathcal{A} be an operator ideal. The following are in order*

(1) *Since \mathcal{K} is surjective, we have*

$$(\mathcal{K} \circ \mathcal{A})^{sur} = \mathcal{K} \circ \mathcal{A}^{sur}.$$

[Theorem 3.6]

(1') *If \mathcal{A} has the l_1 -lifting property, then we have*

$$\mathcal{K} \circ \mathcal{A}^{sur} = (\overline{\mathcal{F}} \circ \mathcal{A})^{sur}.$$

[Theorem 4.5]

(2) Since \mathcal{K} is surjective and also has the l_1 -lifting property, we have

$$\mathcal{A}^{sur} \circ \mathcal{K} = (\mathcal{A} \circ \mathcal{K})^{sur}.$$

[Theorem 4.5]

(2') In particular, since $\mathcal{K} = \overline{\mathcal{F}}^{sur}$ and $\overline{\mathcal{F}}$ has the l_1 -lifting property, we have

$$\mathcal{A}^{sur} \circ \mathcal{K} = \mathcal{A}^{sur} \circ \overline{\mathcal{F}}^{sur} = (\mathcal{A} \circ \overline{\mathcal{F}})^{sur}.$$

(3) It follows from 1 and 2 above, that

$$(\mathcal{A}^{com})^{sur} = (\mathcal{A}^{sur})^{com}.$$

(3') If \mathcal{A} has the l_1 -lifting property, then by 1' and 2' above, we have

$$(\mathcal{A}^{sur})^{com} = (\overline{\mathcal{F}} \circ \mathcal{A})^{sur} \circ \mathcal{K} = (\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}})^{sur} = (\mathcal{A}^{min})^{sur}.$$

Let us rewrite the above results for accessible operator ideals.

Corollary 5.3 *Let \mathcal{A} be an operator ideal. Then*

(a₁) *If \mathcal{A} is left accessible, then*

$$(\mathcal{A}^{com})^{inj} = (\mathcal{A}^{min})^{inj} = \mathcal{K} \circ \mathcal{A}^{inj}.$$

(a₂) *If \mathcal{A} is right accessible and also has the l_∞ -extension property, then we have*

$$(\mathcal{A}^{com})^{inj} = (\mathcal{A}^{min})^{inj} = \mathcal{A}^{inj} \circ \mathcal{K}.$$

(b₁) *If \mathcal{A} is right accessible, then*

$$(\mathcal{A}^{com})^{sur} = (\mathcal{A}^{min})^{sur} = \mathcal{A}^{sur} \circ \mathcal{K}.$$

(b₂) *If \mathcal{A} is left accessible and also has the l_1 -lifting property, then we have*

$$(\mathcal{A}^{com})^{sur} = (\mathcal{A}^{min})^{sur} = \mathcal{K} \circ \mathcal{A}^{sur}.$$

Proof. Since \mathcal{A} is left accessible, $\overline{\mathcal{F}} \circ \mathcal{A} = \mathcal{A}^{min} \subset \mathcal{A}^{com}$. Then by Proposition 5.1(2'), we have

$$\mathcal{K} \circ \mathcal{A}^{inj} = (\overline{\mathcal{F}} \circ \mathcal{A})^{inj} = (\mathcal{A}^{min})^{inj} \subset (\mathcal{A}^{com})^{inj}.$$

Since $\mathcal{A}^{com} \subset \mathcal{K} \circ \mathcal{A} \subset \mathcal{K} \circ \mathcal{A}^{inj}$ and since $\mathcal{K} \circ \mathcal{A}^{inj}$ is injective, (a₁) follows. Dualising the arguments (b₁) also follows. The remaining statements (a₂) and (b₂) follow directly from Propositions 5.1(c') and 5.2(c') respectively. \triangle

The case of the injective-surjective hull is much simpler as we see now.

Proposition 5.4 $(\mathcal{A}^{com})^{inj\ sur} = (\mathcal{A}^{min})^{inj\ sur}$.

Proof. Let $T \in (\mathcal{A}^{com})^{inj\ sur}(X, Y)$, for any Banach spaces X and Y . Then $i_Y \circ Tq_X \in \mathcal{A}^{com}(L_1(B_X), l_\infty(B_{Y^*}))$. Thus there are Banach spaces Z and W and operators $V \in \mathcal{K}(l_1(B_X), Z)$, $U \in \mathcal{K}(W, l_\infty(B_{Y^*}))$ and $S \in \mathcal{A}(Z, W)$ such that $i_Y \circ T \circ q_X = U \circ S \circ V$. Since the Banach spaces $(l_1(B_X))^*$ and $l_\infty(B_{Y^*})$ have the approximation property, we obtain $V \in \overline{\mathcal{F}}(l_1(B_X), Z)$ and $U \in \overline{\mathcal{F}}$. Thus $i_Y \circ T \circ q_X \in (\overline{\mathcal{F}} \circ \mathcal{A} \circ \overline{\mathcal{F}})(l_1(B_X), l_\infty(B_{Y^*}))$ and consequently, $T \in (\mathcal{A}^{min})^{inj\ sur}$. Hence $(\mathcal{A}^{com})^{inj\ sur} \subset (\mathcal{A}^{min})^{inj\ sur}$. As $\mathcal{A}^{min} \subset \mathcal{A}^{com}$, the result follows. \triangle

6 Approximating $\mathcal{A} \circ \mathcal{K}$ and $\mathcal{K} \circ \mathcal{A}$ by finite rank operators

A Banach space X is said to have the approximation property if given a compact set $K \subset X$ and an $\epsilon > 0$, there is a finite rank operator T on X such that $\|Tx - x\| < \epsilon$, for all $x \in K$. Recall that Grothendieck [3] showed that the following statements are equivalent:

- (1) The Banach space X has the approximation property.
- (2) For every Banach space Y , the finite rank operators $\mathcal{F}(Y, X)$ is dense in $\mathcal{B}(Y, X)$ in the topology of uniform convergence on compact sets.
- (3) For every Banach space Y , the finite rank operators $\mathcal{F}(X, Y)$ is dense in $\mathcal{B}(X, Y)$ in the topology of uniform convergence on compact sets.
- (4) For every Banach space Y , the finite rank operators $\mathcal{F}(Y, X)$ is dense in compact operators $\mathcal{K}(Y, X)$ in the operator norm.

For the approximation property in the dual of a Banach space we have [4, Theorem 1.e.5]

- (5) The dual X^* of a Banach space X has the approximation property if and only if for every Banach space Y , the finite rank operators $\mathcal{F}(X, Y)$ is dense in compact operators $\mathcal{K}(X, Y)$ in the operator norm.

The authors in [7, Definitions 2.1, 2.2 and 6.1] introduced the notions of a p -compact set, a p -compact operator and the corresponding notion of the p -approximation property, which is to approximate the identity operator by finite rank operators on p -compact sets. However, the p -approximation property is equivalent to the density of finite rank operators in the ideal of p -compact operators \mathcal{K}_p in a norm weaker than the ideal norm of \mathcal{K}_p [7, Theorem 6.3]. Thus a prototype of the above equivalence (1) \Leftrightarrow (4), for

$1 \leq p < \infty$, could not be achieved. Later, the authors [8, Definition 4.4], taking a cue from (2) above, introduced another notion, namely, the approximation property of type p , for $1 \leq p \leq \infty$, in terms of a locally convex topology λ_p on the operator ideal Π_p . It was proved [8, Theorems 4.5 and 4.6] that both (2) \Leftrightarrow (4) and (5) of the above list have suitable prototypes for the approximation property of type p . To be precise, a Banach space X has the approximation property of type p if and only if the finite rank operators are dense in $\mathcal{K}_p^{dual}(Y, X)$ for all Banach spaces Y , and the dual space X^* has the approximation property of type p if and only if the finite rank operators are dense in $\mathcal{K}_p^{dual\ dual}(X, Y)$ for all Banach spaces Y . In what follows, we seek to reinvent the later for a general operator ideal, and in particular, for injective and surjective operator ideals.

First, we record an observation due to Grothendieck, that is essentially contained in his proof of the several equivalent formulations of the approximation property [4, Theorem 1.e.4 (4) \Rightarrow (1)].

Lemma 6.1 *Let $(X, \|\cdot\|)$ be a Banach space and K be a compact subset of X . Then there is a Banach space $(Y, \|\cdot\|_0)$ be formally contained in X such that the formal identity map $i_Y : Y \rightarrow X$ is a compact operator with $K \subset \overline{i_Y(BallY)}$. Furthermore, given a continuous function g on Y and $\epsilon > 0$, there exists an $f \in X^*$ such that*

$$|f(y) - g(y)| < \epsilon, \text{ for all } y \in i_Y(BallY).$$

Next, again in the spirit of (2) above, we define a locally convex topology on $\mathcal{A}(X, Y)$ for an operator ideal \mathcal{A} followed by a corresponding approximation property in the following manner.

Definition 6.2 *Let \mathcal{A} be a operator ideal. For Banach spaces X, Y and a compact set $K \subset X$ we define a seminorm $\|\cdot\|_K$ on $\mathcal{A}(X, Y)$ given by*

$$\|T\|_K = \inf\{\alpha_\kappa(T \circ i_Z) : i_Z : Z \rightarrow X \text{ as above}\},$$

where $\alpha_\kappa(T) = \inf\{\alpha(T_1)\|T_2\| : T = T_1 \circ T_2 \in \mathcal{A} \circ \mathcal{K}\}$ is the usual composition norm on $\mathcal{A} \circ \mathcal{K}$. Then the family of seminorms $\{\|\cdot\|_K : K \subset X \text{ is compact}\}$ defines a locally convex topology $\lambda_{\mathcal{A}}$ on $\mathcal{A} \circ \mathcal{K}(X, Y)$.

Definition 6.3 *Let \mathcal{A} be a operator ideal. A Banach spaces X is said to have the \mathcal{A} -approximation property (\mathcal{A} -a.p., for short) if for every Banach space Y , $\mathcal{F}(Y, X)$ is dense in $\mathcal{A}(Y, X)$ in the $\lambda_{\mathcal{A}}$ -topology.*

Note that for the operator ideal \mathcal{B} , the topology $\lambda_{\mathcal{B}}$ is the topology on $\mathcal{B}(X, Y)$ of uniform convergence on compact sets of X . Recently, the authors [8, Definition 4.3] have defined for each $1 \leq p \leq \infty$, a locally convex topology λ_p on $\Pi_p(X, Y)$. In terms of the Definition 6.2 above this is the Π_p -topology on $\Pi_p(X, Y)$. A Banach space X is said to have the approximation property of *type p* [8, Definition 4.4] if for any Banach space Y , the finite rank operators $\mathcal{F}(Y, X)$ is dense in $\Pi_p(Y, X)$ in the λ_p -topology. Thus in terms of the Definition 6.3 above this is the Π_p -approximation property on X .

Now, we characterise the \mathcal{A} -approximation property in terms of the density of finite rank operators \mathcal{F} in $\mathcal{A} \circ \mathcal{K}$ in its composition operator norm α_{κ} .

Theorem 6.4 *Let \mathcal{A} be an operator ideal. A Banach space X has the \mathcal{A} -approximation property if and only if for every Banach space Y , we have*

$$\mathcal{A} \circ \mathcal{K}(Y, X) \subset \overline{\mathcal{F}(Y, X)}^{\alpha_{\kappa}}.$$

Proof. Let $T \in \mathcal{A} \circ \mathcal{K}(Y, X)$ and $\epsilon > 0$. Consider a factorisation $T = T_1 \circ T_2$ where $T_1 \in \mathcal{A}(G, X)$ and $T_2 \in \mathcal{K}(Y, G)$ for some Banach space G . Set $K = \overline{T_2(\text{Ball}Y)}$. Since X has the \mathcal{A} -a.p., there is a $S_0 \in \mathcal{F}(G, X)$ such that

$$\|T_1 - S_0\| < \epsilon/\|T_2\|.$$

Furthermore, as in the above Lemma 6.1, we can find a Banach space Z formally contained in G such that $i_Z : Z \rightarrow G$ is compact and we have

$$\alpha_{\kappa}((T_1 - S_0) \circ i_Z) < \epsilon/\|T_2\|.$$

Since $i_Z(k) = k$ for all $k \in K = \overline{T_2(\text{Ball}Y)}$, we get $i_Z \circ T_2 = T_2$. Set $S = S_0 \circ T_2$. Then $S \in \mathcal{F}(Y, X)$ and we have

$$\alpha_{\kappa}(T - S) = \alpha_{\kappa}(T_1 \circ T_2 - S_0 \circ T_2) \leq \alpha_{\kappa}((T_1 - S_0)i_Z)\|T_2\| < \epsilon.$$

Conversely, let $T \in \mathcal{A}(Y, X)$, $K \subset Y$ a compact set and $\epsilon > 0$. By Lemma 6.1, there is a Banach space Z formally contained in Y such that $i_Z : Z \rightarrow Y$ is compact and $K \subset i_Z(\text{Ball}Z)$. Then $T \circ i_Z \in \mathcal{A} \circ \mathcal{K}(Z, X)$ so that there is an $S_1 \in \mathcal{F}(Z, X)$ such that $\alpha_{\kappa}(T \circ i_Z - S_1) < \epsilon/2$. Consider a factorisation $S_1 = S_2 \circ S_3$ for some operators $S_2 \in \mathcal{F}(G, X)$, $S_3 \in \mathcal{F}(Z, G)$ and some Banach space G . Choose $S_4 \in \mathcal{F}(Z, G)$ such that $\|S_3 - S_4\| < \epsilon/4\alpha_{\kappa}(S_2)$. Let $S_4 \sim \sum_{i=1}^n f_i \otimes x_i$, with $f_1, f_2, \dots, f_n \in Z^*$ and $x_1, x_2, \dots, x_n \in G$. Again by Lemma

6.1, given $\delta > 0$ we can find $g_1, g_2, \dots, g_n \in Y^*$ such that $|f_i(z) - g_i(z)| < \delta$ for all $z \in \text{Ball} Z$. Indeed, we may choose $\delta = \epsilon/4n\alpha_\kappa(S_2) \max_{1 \leq i \leq n} \|x_i\|$ such that $\|S_4 - S'\| < \epsilon/4\alpha_\kappa(S_2)$, where $S' = \sum_{i=1}^n g_i \cdot i_Z \otimes x_i$. Set $S'' = \sum_{i=1}^n g_i \otimes x_i$ so that $S'' \in \mathcal{F}(Y, G)$ and $S' = S'' \cdot i_Z$. Thus $S = S_2 \cdot S'' \in \mathcal{F}(Y, X)$ and

$$\begin{aligned} \alpha_\kappa((T - S) \cdot i_Z) &\leq \alpha_\kappa(T \cdot i_Z - S_1) + \alpha_\kappa(S_1 - S_2 \cdot S'' \cdot i_Z) \\ &\leq \epsilon/2 + \alpha_\kappa(S_2 \cdot S_3 - S_2 \cdot S') \\ &\leq \epsilon/2 + \alpha_\kappa(S_2) \|S_3 - S'\| \\ &\leq \epsilon/2 + \alpha_\kappa(S_2) \{\|S_3 - S_4\| + \|S_4 - S'\|\} < \epsilon. \end{aligned}$$

It follows that $\|T - S\|_K \leq \alpha_\kappa((T - S) \cdot i_Z) < \epsilon$, so that X has the \mathcal{A} -a.p. \triangle

It was proved by the authors [8, Theorem 4.5] that a Banach space X has the approximation property of *type* p if and only if for every Banach space Y , the finite rank operators $\mathcal{F}(Y, X)$ is dense in the composition operator ideal $\Pi_p \cdot \mathcal{K}(Y, X)$ in its factorisation norm $\pi_{p\kappa}$. Thus, Theorem 4.5 in [8] is a special case of Theorem 6.4 above, for the operator ideal Π_p .

Theorem 6.5 *Let \mathcal{A} be an operator ideal and X a Banach space. If the dual space X^* has the \mathcal{A} -approximation property then for every Banach space Y we have*

$$\mathcal{K} \cdot \mathcal{A}^{dual}(X, Y) \subset \overline{\mathcal{F}(X, Y)}^{\kappa\alpha^d}.$$

The converse holds if, in addition, $\mathcal{A} \subset \mathcal{A}^{dual\ dual}$ (equivalently, $\mathcal{A}^{dual\ dual} = \mathcal{A}^{reg}$).

Proof. Let X^* have the \mathcal{A} -a.p. Then by Theorem 6.4 above, for every Banach space Y , we have

$$\mathcal{A} \circ \mathcal{K}(Y, X^*) \subset \overline{\mathcal{F}(Y, X^*)}^{\alpha_\kappa}.$$

It follows that $\mathcal{A} \circ \mathcal{K}(Y^*, X^*) \subset \overline{\mathcal{F}(Y^*, X^*)}^{\alpha_\kappa}$, for all Banach spaces Y . Let $T \in \mathcal{K} \cdot \mathcal{A}^{dual}(X, Y)$ and $\epsilon > 0$. Then there is a Banach space G and operators $T_1 \in \mathcal{K}(G, Y)$ and $T_2 \in \mathcal{A}^{dual} X, G$ such that $T = T_1 \circ T_2$. Then $T^* = T_2^* \circ T_1^* \in \mathcal{A} \circ \mathcal{K}(Y^*, X^*)$ so that there is an $S \in \mathcal{F}(Y^*, X^*)$ with $\alpha_\kappa(T^* - S) < \epsilon$. We may write $T^* - S = S_1 \circ S_2$, where $S_1 \in \mathcal{A}(Z, X^*)$ and

$S_2 \in \mathcal{K}(Y^*, Z)$ for some Banach space Z such that $\alpha(S_1) < \epsilon$ and $\|S_1\| = 1$. Now, $S^* \in \mathcal{F}(X^{**}, Y)$ and

$$\begin{aligned} {}_{\kappa}\alpha^d(T - S^* \circ \sigma_X) &= {}_{\kappa}\alpha^d(T^{**} \circ \sigma_X - S^* \circ \sigma_X) \\ &\leq {}_{\kappa}\alpha^d(T^{**} - S^*)\|\sigma_X\| \\ &\leq \|S_2^*\|\alpha^d(S_1^*) \\ &= \alpha^{**}(S_1) \leq \alpha(S_1) < \epsilon, \end{aligned}$$

where $\sigma_X : X \rightarrow X^{**}$ is the canonical embedding.

Conversely, let \mathcal{A} be an operator ideal satisfying $\mathcal{A} \subset \mathcal{A}^{dual\ dual}$, $T \in \mathcal{A} \circ \mathcal{K}(Y, X^*)$ and $\epsilon > 0$. Then there is a Banach space G and operators $T_1 \in \mathcal{A}(G, X^*)$ and $T_2 \in \mathcal{K}(Y, G)$ such that $T = T_1 \circ T_2$. It follows that $T^* \in \mathcal{K} \circ \mathcal{A}^{dual}(X^{**}, Y^*)$, so that $T^* \circ \sigma_X \in \mathcal{K} \circ \mathcal{A}^{dual}(X, Y^*)$, and by our assumption there is an $S \in \overline{\mathcal{F}}(X, Y^*)$ such that ${}_{\kappa}\alpha^d(T^* \circ \sigma_X - S) < \epsilon$. Now we can choose a Banach space Z and operators $S_1 \in \mathcal{K}(Z, Y^*)$ and $S_2 \in \mathcal{A}^{dual}(X, Z)$ such that $T^* \circ \sigma_X - S = S_1 \circ S_2$ with $\|S_1\| = 1$ and $\alpha^d(S_2) < \epsilon$. Thus $S^* \in \overline{\mathcal{F}}(Y^{**}, X^*)$ and we have

$$\begin{aligned} \alpha_{\kappa}(T - S^* \circ \sigma_Y) &= \alpha_{\kappa}(\sigma_X^* T^{**} \circ \sigma_Y - S^* \sigma_Y) \\ &= \alpha_{\kappa}(S_2^* \circ S_1^* \circ \sigma_Y) \\ &\leq \alpha(S_1^*)\|S_1^*\|\|\sigma_Y\| \\ &= \alpha^d(S_2) < \epsilon. \end{aligned}$$

It follows that X^* has the \mathcal{A} -a.p. \triangle

The authors in [8, Theorem 4.6] have shown that the dual X^* of a Banach space X has the approximation property of *type p* if and only if $\mathcal{F}(X, Y)$ is dense in $\mathcal{K} \cdot \Pi_p^{dual}(X, Y)$ in its composition ideal norm for all Banach spaces Y . Thus, Theorem 4.6 in [8] is a special case of Theorem 6.5 above.

Finally, at the end we discuss certain special cases of the \mathcal{A} -a.p. for injective and surjective operator ideals \mathcal{A} with ℓ_{∞} -extension or ℓ_1 -lifting property.

Corollary 6.6 *Let \mathcal{A} be a injective operator ideal with the ℓ_{∞} -extension property and X a Banach space. Then we have the following:*

(a) *X has the \mathcal{A} -approximation property if and only if*

$$(\mathcal{A}^{min})^{inj}(Y, X) = \mathcal{A}^{min}(Y, X),$$

for all Banach spaces Y .

(b) The dual space X^* has the \mathcal{A} -approximation property if and only if

$$((\mathcal{A}^{dual})^{min})^{sur}(X, Y) = (\mathcal{A}^{dual})^{min}(X, Y),$$

for all Banach spaces Y .

Proof. (a) Since \mathcal{A} is injective, it is right accessible and it also has the ℓ_∞ -extension property. Thus, by Corollary 5.3(a_2) we have $(\mathcal{A}^{min})^{inj} = \mathcal{A}^{inj} \circ \mathcal{K} = \mathcal{A} \circ \mathcal{K}$. By Theorem 6.4, X has the \mathcal{A} -a.p. if and only if $\mathcal{A}^{min}(Y, X)$ is dense in $\mathcal{A} \circ \mathcal{K}(Y, X)$ for all Banach spaces Y . Thus X has the \mathcal{A} -a.p. if and only if $(\mathcal{A}^{min})^{inj}(Y, X) = \mathcal{A}^{min}(Y, X)$, for all Banach spaces Y .

(b) Since \mathcal{A}^{dual} is surjective, it is left asseccible and by Theorem 4.9 it also has the ℓ_1 -lifting property. Thus by Corollary 5.3(b_2) we have $((\mathcal{A}^{dual})^{min})^{sur} = (\mathcal{A}^{dual})^{min}$. Now, by Theorem 6.5, X^* has the \mathcal{A} -a.p. if and only if $(\mathcal{A}^{dual})^{min}(X, Y)$ is dense in $\mathcal{K} \circ \mathcal{A}^{dual}(X, Y)$ for all Banach spaces Y . Thus X^* has the \mathcal{A} -a.p. if and only if $((\mathcal{A}^{dual})^{min})^{sur}(X, Y) = (\mathcal{A}^{dual})^{min}(X, Y)$, for all Banach spaces Y . \triangle

Again note that since Π_p is injective and has the ℓ_∞ -extension property, Theorems 4.5 and 4.6 in [8] are special cases of the above corollary. Let us also note here that we do not know whether there is an injective operator ideal that fails the ℓ_∞ -extension property.

Corollary 6.7 *Let \mathcal{A} be a right accessible surjective operator ideal and X be a Banach space. Then we have the following:*

(a) X has the \mathcal{A} -approximation property if and only if

$$(\mathcal{A}^{min})^{sur}(Y, X) = \mathcal{A}^{min}(Y, X),$$

for all Banach spaces Y .

(b) The dual space X^* has the \mathcal{A} -approximation property if and only if

$$((\mathcal{A}^{dual})^{min})^{inj}(X, Y) = (\mathcal{A}^{dual})^{min}(X, Y),$$

for all Banach spaces Y .

Proof. (a) By Corollary 5.3(b_1), we have $(\mathcal{A}^{min})^{sur} = \mathcal{A} \circ \mathcal{K}$. Now it follows by Theorem 6.4 that X has the \mathcal{A} -a.p. if and only if $(\mathcal{A}^{min})^{sur}(Y, X) = \mathcal{A}^{min}(Y, X)$, for all Banach spaces Y .

(b) Since \mathcal{A}^{dual} is injective, by Corollary 5.3(a_1), we have $((\mathcal{A}^{dual})^{min})^{inj} = \mathcal{K} \circ \mathcal{A}^{dual}$. Now it follows by Theorem 6.5 that X^* has the \mathcal{A} -a.p. if and only if $((\mathcal{A}^{dual})^{min})^{inj}(X, Y) = (\mathcal{A}^{dual})^{min}(X, Y)$, for all Banach spaces Y . \triangle

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